The Consequences of Ramanujan's Infinite sum on Number Lines and Coordinate Systems

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***Abstract*—A theoretical and geometrical explanation of infinities and infinitesimals can be justified by interpreting Ramanujan's sum of all natural numbers along with the Riemann Zeta function geometrically, and simultaneously offering a method of visualizing infinities and infinitesimal quantities. Another result of the geometric manipulation is the ability to quantize continuity to some extent, as complete quantization of continuity is impossible. All of this is possible due to the closed shape of a coordinate system composed of *n* dimensions, as those *n* dimensions show *n*+1 dimensional behaviour at infinite scales. This concept can be extrapolated to the dimension of infinitesimals, as they show the same properties with respect to integers as integers do with respect to infinities. Upon further extrapolation to it is possible to quantize infinitesimal quantities and allow transformations between the dimensions of infinitesimals and integers. In this way it has been proved that a system of coordinates contains infinite sub-dimensions and infinite super-dimensions.**

I. Introduction

Riemann's Zeta function can only be understood graphically when we extend its domain to numbers smaller than one, and simultaneously enforcing a constraint that all points in the domain and resulting range must be differentiable. The need of an argand plane to do this removes the physical intuition of adding we have always known. A method has been suggested which preserves the original intuition of adding numbers in the number line, while simultaneously providing a geometrical solution to the infinite sums that contain numbers greater than 1.

Calculus as a tool allows us to understand the nature of functions to a great extent. Although it is completely based on the concept of infinitesimally small and infinitely large quantities, it fails to provide any principle or justification for them as individual entities in mathematics, and instead treats them as tools for evaluation. Quantities beyond our comprehension are usually dubbed undeterminable quantities. These quantities can be realized to some extent using the system explained in the following sections.

II. Analysis of the Problem

The one dimensional number line requires only one variable to depict any number in that space.

We can identify the set of numbers in our number system with a corresponding set (Real, Rational, Whole, etc.). In order to keep the things simple, we will only deal with integers. We know that the number line spans all real numbers from -∞ to +∞. Upon further inspection of the behaviour of the number line, we can conclude that +∞ is a "very large positive number" and -∞ is a "very large negative number". But if we look at Ramanujan's sum of all natural numbers, i.e., Riemann's Zeta function with an input of (-1):

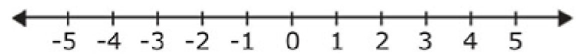

Fig. 1. The Linear Number line

$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = 1+2+3+4+5... = -\frac{1}{12}.$$

It seems nonsensical at first. Usually, it makes sense to say that $+\infty$ is a very large number, since adding all the numbers upto that "very large number" should give us an even larger number. But something completely unexpected happens. It seems like we go *backward in the number line* instead of forward, which is very much unlike what normally happens when numbers are added. This anomaly in the number system proves that our understanding of the geometry of addition and geometry of the number line is incoherent. Both cannot simultaneously be true.

Another observation is that the backward anomaly only comes into play when we start dealing with *infinite* quantities. This means that the anomaly takes place at infinite scales. Therefore the solution must also be able to coherently perform arithmetic operations at infinite and infinitesimal scales.

A new system of coordinates must be arrived at that could explain how traversing forward in the system (addition) could leave you further back from where you started. This can only be done by completely altering the geometry of the current one-dimensional coordinate system we use, i.e., number line.

III. A Two Dimensional Solution to a One Dimensional Problem

The solution to this problem can only be found if we zoom out of the number line *infinitely*. If we think of the number line as a part of a circle of an infinitely large radius with each point on the new number line infinitely far from the center, our problem can be solved. We do not care about where the center is at, only that it exists at an infinitely long distance away from the origin. So, if we perform this infinite zoom, we should hypothetically observe something like this:

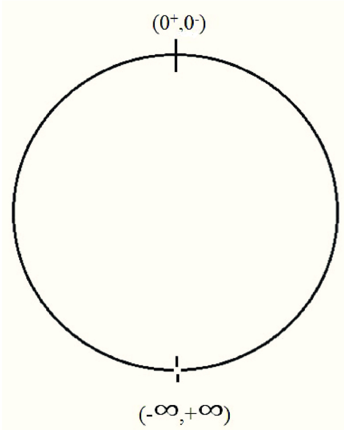

Fig. 2. The Circular number line

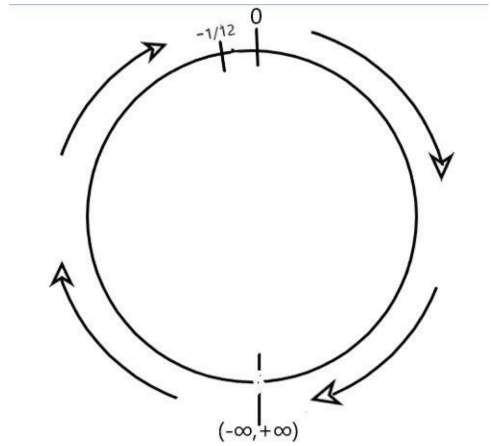

Fig. 4. Rotational path of Ramanujan's sum in the Circular number line.

The coordinate (0^{-},0^{+}) are not to be confused with the (*x*,*y*) cartesian coordinates. The (0^{+},0^{-}) here suggests that we are approaching zero from both sides of the new-found circle coordinate system, just like the old and anomalous number line. The only difference is, this time we also have an opposite (-∞,+∞). But instead of approaching a number like its zero counterpart, the infinities get *infinitesimally close to a discontinuity*. Therefore the circle isn't actually complete; There is a discontinuity at one end. This discontinuity keeps -∞ and +∞ infinitesimally apart from each other.

Remark: The reason for using circular shape for the 1-dimensional coordinate system instead of any other symmetrical 2-D figure was because the circle has the most symmetry out of any other shape.

IV. Explaining Ramanujan's Sum Geometrically

We will define the rules of this new circular coordinate system according to the results we obtain out of the Riemann zeta function. The advantage of this particular system of coordinates is that it can explain why the sum of all natural numbers is negative, i.e., how traversing forward in the number line could leave you further behind from where you started.

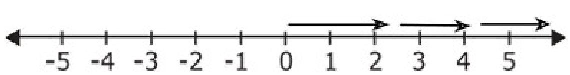

Fig. 3. Path of Addition in the Linear number line

Since the new system is circular, numbers are forced to travel a curvilinear path in the process of addition, instead of the old rectilinear path. Therefore, when we perform Ramanujan's infinite sum in the new coordinate system, it is geometrically interpreted in Figure 4.

The circular path is defined in such a way that it allows us to travel *backward* in the coordinate system, hence giving us the specific result we require: -1/12.

V. Extrapolating the Solution to n Dimensions

We can extrapolate this transformation to the cartesian coordinate system as well. The regular cartesian coordinate system composes of coordinates of the form (x, y) where $x, y \in \mathbb{R}$.

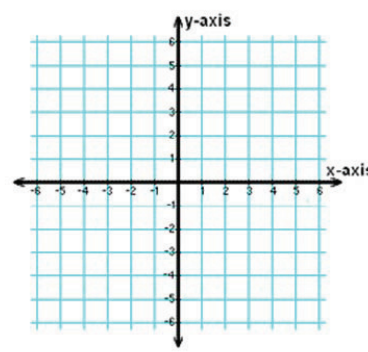

Fig. 5. The Cartesian Coordinate system

In order to transform this coordinate system such that it too, allows circular summation, it has to be in the form of sphere with its radius infinitely far from the origin, and its infinities diverging into a shared discontinuity. The system would hypothetically appear like a spherical ball shown in Figure 6.

If we do the same transformation to the 3-D coordinate system, we end up with a hypersphere.

VI. The Consequences: Impact on Definite Integrals

Since the new cartesian system is spherical in nature, it must possess Gaussian curvature. The positive Gaussian curvature of the new spherical cartesian system is infinitesimally small, due to its infinitely large radius. This means that all areas calculated in the two dimensional cartesian system are approximations:

$$\int_S dA > \int_a^b f(x)dx.$$

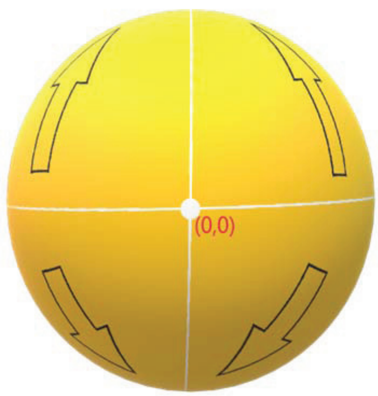

Fig. 6. The Spherical Cartesian Coordinate system

Where S is the areal surface corresponding to the function on the spherical coordinate system. This means calculating any integral between two limits (*a,b*) is an approximation of the respective surface integral about the spherical coordinate system, since the curvature adds more area.

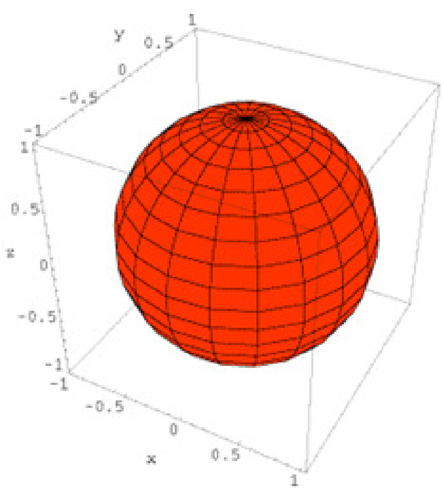

Fig. 7. Positive Gaussian curvature of a sphere

But the approximation is actually a really good one, since the Gaussian curvature of the sphere of infinitely large radius is infinitesimally small and is of no significance when calculating the value of an integral. Similarly, the surface integral in a locally 3-dimensional coordinate system would just be an approximation of the respective volume integral about the 4-dimensional hyperspherical coordinate system. This extrapolation can be carried out for any coordinate system that is locally *n*-dimensional.

VII. The Creation of a New Coordinate Systems

We can apply the concept of making of infinities converge into a discontinuity to within our regular number line. If we take all the real numbers between 0 and 1, and choose a central origin, which would be the arithmetic mean of the two boundary numbers (0 and 1 in this case), we would get a *coordinate system of infinitesimals*:

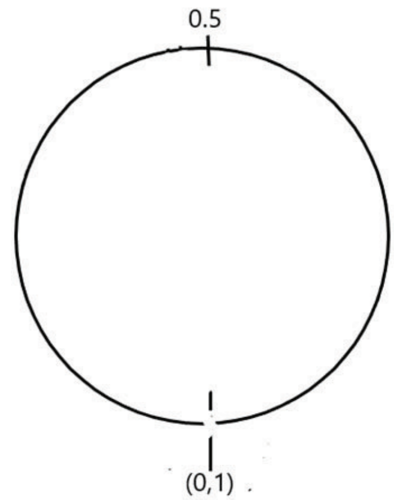

Fig. 8. The Infinitesimal Coordinate system

Analogous to Figure 2, we would have a coordinate system with its origin at 1.5, the arithmetic mean of the infinities. The central origin 0.5 is *infinitely far away from 0 and 1 in the new coordinate system*. This means that the new coordinate system does not have the same 'infinity' as our circular number system . The so called 'infinities' - 0 and 1 would also converge, analogous to the number line, into a discontinuity. Note that the pseudo-infinities 0 and 1 never overlap. They appear to converge into a discontinuity, and therefore when we transform back to the regular number system, 0 and 1 retain their original places. The numbers between 0.5 and 1 would be 'infinitesimals', and the numbers between 0 and 0.5 being the 'negative' counterparts of those infinitesimals. If α is an infinitesimal in this coordinate system,

$$\alpha - \alpha = 0.5.$$

This means that rules of arithmetic operations completely change when we undergo this coordinate system transformation. The laws of this new coordinate system can be decided by cross-checking with the regular coordinate system.

VIII. Partial Quantization of Continuity

Upon careful inspection of such transformed coordinate systems, we observe that the continuous coordinate system is actually being *quantized into discrete infinitesimal quantities*. This means that all that was considered indeterminate infinitesimal in the regular coordinate system, is now defined to be entity with properties analogous to that of an integer, in the new coordinate system.

This means that the calculus of the regular coordinate system can be converted into algebra of the infinitesimal coordinate system. Therefore, in a cartesian system the integral:

$$\int_0^a f(x)\ dx$$

can be converted to an infinite sum:

$$\sum_{x=0}^{a} H(f(x)) = \sum_{x=0}^{a} \Gamma(x);$$

which occurs in the infinitesimal coordinate system, where H is the *system operator* which transforms the regular function $f(x)$ into the infinitesimal function $\Gamma(x)$.

This means any unsolvable definite (proper) integral can be converted into an infinite sum that may be solvable assuming we know three things:

- The coordinate system the integral is performing in.
- The coordinate system we want the infinite sum to be performed in.
- The system operator that allows us to transition between the two said coordinate systems.

If the above three requirements are met, we can solve any non elementary proper integral.

IX. Advantages and Conclusions

- Ramanujan's infinite sum is very counter intuitive, since it suggests that summing up infinite numbers leads us further back than where we started.
- The intuitive answer to such an anomaly is that the ends of the coordinate system are somehow joined (but discontinuous). Maintaining circular geometry seems to be the most intuitive solution.
- Upon extrapolating this concept to higher dimensions, starting with two dimensions, we apply the same type of transformation, we obtain a spherical coordinate system.
- This means that all locally *n* dimensional coordinate systems show *n+1* dimensional behaviour at infinite scales. We can see that the previously simple cartesian coordinate system isn't as simple as it seems. It is a spherical coordinate system with infinitesimal positive Gaussian curvature, composed of infinite sub-infinities and super-infinities.
- Since this transformation has a topological effect, we can also conclude that any definite integrals performed in the previously planar cartesian system is actually an approximation in the new spherical cartesian system.
- This is due to the fact that a sphere has positive Gaussian curvature and this requires that there be more area under a curve on a spherical surface as compared to a flat surface.
- The approximation, however is nearly exact, since the Gaussian curvature of the sphere is so infinitesimally small, it has no real significance.
- Another consequence of this system transformation is that we can partially quantize continuity. We create a new coordinate system with seperate laws of arithmetic.
- This would allow us to transform between coordinate systems, given that we know of the system operator.
- Using a system operator we can transform any function in one coordinate system into a function in another coordinate system. This tool can prove useful for computing many integrals and sums.

Compliance with ethical standards

Conflict of interest: Authors declare that they have no conflict of interest.
Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Acknowledgment

I would like to thank my advisor and guide, Dr. Rakesh Bajaj for his invaluable help and guidance with my amatuer attempt at mathematical research. I would like to thank my school, Gurukul The School and my Principal, Mr. Gaurav Bedi, for connecting me with people of mathematical background. I could not have made it this far without the help of all my teachers, especially Mr. Shambik sir and Mr. Vikas sir, who have always been there to listen to my ideas and question them, something very few people do.

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